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Orthogonal Polynomials with Discrete Spectra on the Real Line

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Orthogonal polynomials defined by the classical three-term recurrence relation are considered. Conditions on the coefficients are obtained which are sufficient for the spectrum to be a denumerable set which is unbounded above and below. Observations are made concerning extremal solutions and the accumulation points of the zeros of the orthogonal polynomials when the corresponding Hamburger moment problem is indeterminate.

1. INTRODUCTION

Let $\{P_n(x)\}$ be a sequence of monic polynomials satisfying a three term recurrence relation of the form

$$\begin{gathered} P_n(x) = (x - c_n)\, P_{n-1}(x) - \lambda_n P_{n-2}(x), \qquad n = 1, 2, 3,..., \\ P_{-1}(x) = 0, \;\; P_0(x) = 1; \;\; c_n \text{ real}, \;\; \lambda_n > 0 \;\; (n \geqslant 1). \end{gathered} \tag{1.1}$$

It is well known that (1.1) is necessary and sufficient for the existence of a real distribution function ψ with an infinite spectrum $S(\psi)$ (support of $d\psi$) with respect to which the polynomials are orthogonal,

As was already known to Stieltjes, $S(\psi)$ is a bounded set if and only if $\{c_n\}$ and $\{\lambda_n\}$ are both bounded sequences. Stieltjes [21] also proved that if $c_n = 0$ $(n > 0)$ and $\{\lambda_n\}$ converges to 0, then $S(\psi)$ is a denumerable set with 0 as its only point of accumulation. Krein [1, p. 230] gave a remarkable generalization of Stieltjes' theorem in which he completely characterized all cases for which $S(\psi)$ is a bounded denumerable set whose derived set is an arbitrarily prescribed finite set. At the opposite extreme, an early result of Blumenthal [2] says that if $\{c_n\}$ and $\{\lambda_n\}$ both converge to finite limits c and λ, respectively, then the zeros of the $P_n(x)$ are dense in the interval $(\sigma, \tau) = (c - 2\sqrt{\lambda}, c + 2\sqrt{\lambda})$ and $S(\psi)$ has at most denumerably many points on the

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complement, with σ and τ the only possible accumulation points. (Proofs of these can also be found in [8].)

Some more recent examples of conditions yielding results concerning denumerable spectra can be found in [5, 7, 8, 9, 15]. Nevai and others (see, e.g. [4, 12, 13, 17, 18, 19]) have dealt with the case covered by Blumenthal's theorem. Very recently, Máté and Nevai [16] have found rather general conditions that yield absolute continuity of ψ. When $\{c_n\}$ and $\{\lambda_n\}$ are not both bounded, much less is known. If the sequences are restricted so the spectrum is contained in a half line, some conditions are known that guarantee that $S(\psi)$ is a denumerable set with no finite accumulation point (e.g. [8, 11]). There is also an analog of Blumenthal's theorem ([5, 8]).

When $S(\psi)$ extends over $(-\infty, \infty)$, however, we know of no theorems that yield additional information about the spectrum except in the symmetric case: $P_n(-x) = (-1)^n P_n(x)$. Indeed, the only explicitly known examples for this case are the Meixner polynomials of the second kind and Pollaczek's generalization (see [8, pp. 179, 186]). As a first step toward filling in this void, we present two theorems that apply when the spectrum is unbounded at both ends and yield the conclusion that the spectrum is a denumerable set. We also make some observations concerning the existence of "best" extremal solutions when the Hamburger moment problem is indeterminate.

2. Preliminaries; The Moment Problem

We will be concerned with (1.1) under conditions such that the zeros of the orthogonal polynomials are unbounded both above and below. This is equivalent to the condition that if we we set

$$\alpha_n(t) = \frac{\lambda_{n+1}}{(c_n - t)(c_{n+1} - t)} \tag{2.1}$$

then for every real t, $\{\alpha_n(t)\}$ is not a chain sequence [8, p. 109]. Simple sufficient conditions for this are, for example,

(i) $\inf_n c_n = -\infty$ and $\sup_n c_n = \infty$;

(ii) $\lim_{n\to\infty} c_n = \infty$ and $\liminf_{n\to\infty} \lambda_{n+1}/(c_n c_{n+1}) > \frac{1}{4}$.

We denote the zeros of $P_n(x)$ by $x_{n,i}$:

$$x_{n,1} < x_{n,2} < \cdots < x_{n,n}$$

and denote the set of all $x_{n,i}$ by X. Finally, we denote by σ and τ the smallest and largest limit points (in the extended real number system) of the zeros:

$$\sigma = \lim_{i\to\infty} \lim_{n\to\infty} x_{n,i}, \qquad \tau = \lim_{j\to\infty} \lim_{n\to\infty} x_{n,n-j}. \tag{2.2}$$

Whenever the Hamburger moment problem is determined, σ and τ are also the smallest and largest accumulation points of the spectrum $S(\psi)$ [8].

Since we wish to draw conclusions about the derived set of the spectrum, we will have to take into account the status of the corresponding Hamburger moment problem, especially so in view of Stone's theorem [20, p. 59; 22] which asserts that if the Hmp is indeterminate, there are solutions whose spectra have derived sets which coincide with arbitrarily prescribed closed sets. If the Hmp is indeterminate, there also arises the question of the existence of a "best" extremal solution; that is, one whose spectrum coincides with the set of limit points of the zeros of the orthogonal polynomials. This question is settled in the case where the moment problem can be reduced to a Stieltjes moment problem by a translation [6] and also in the case of a symmetric Hmp (i.e., all moments of odd order vanish) [10]. In the general case, the situation, in one sense, turns out to be surprisingly easy to describe on the basis of well known results from the analytic theory of J-fractions.

We consider the J-fraction corresponding to the recurrence relation (1.1)

$$\frac{\lambda_1 \,|}{|\, z-c_1} - \frac{\lambda_2 \,|}{|\, z-c_2} - \frac{\lambda_3 \,|}{|\, z-c_3} - \cdots. \tag{2.3}$$

The nth convergent (approximant) of (2.3) can then be written

$$K_n(z) = \frac{F_n(z)}{P_n(z)} \tag{2.4}$$

where the numerator polynomials $F_n(x) = P_{n-1}^{(1)}(x)$ satisfy the recurrence (1.1) but with the initial conditions

$$F_{-1}(x) = -1/\lambda_1, \qquad F_0(x) = 0.$$

Now assume that the Hmp associated with (2.3) is indeterminate. We then consider the four sequences of polynomials $A_n(z)$, $B_n(z)$, $C_n(z)$, $D_n(z)$ which can be defined in terms of the $P_n(z)$ and the $F_n(z)$, and which converge, uniformly on compact sets, to entire functions A, B, C, D, respectively, such that

$$A(z)\,D(z) - B(z)\,C(z) = 1, \tag{2.5}$$

and the extremal solutions ψ_s of the moment problem are determined by

$$I_s(z) = \frac{A(z) - sC(z)}{B(z) - sD(z)} = \int_{-\infty}^{\infty} \frac{d\psi_s(t)}{z-t}, \qquad -\infty \leqslant s \leqslant \infty \tag{2.6}$$

(see [20, p. 51]). Further, from the definition of these polynomials, we have

$$P_n(z) = -P_n(0)\, B_n(z) + F_n(0)\, D_n(z), \tag{2.7}$$

$$K_n(z) = \frac{A_n(z) - K_n(0)\, C_n(z)}{B_n(z) - K_n(0)\, D_n(z)}. \tag{2.8}$$

Let $a = \{n_\nu\}_{\nu=1}$ be an increasing sequence of positive integers and consider the corresponding subsequence of orthogonal polynomials, $P_{n_\nu}(x)$. Let X_a denote the corresponding set of zeros:

$$X_a = \{x_{n_\nu,i} : 1 \leqslant i \leqslant n_\nu;\ \nu = 1, 2, 3,...\}. \tag{2.9}$$

Considering the corresponding subsequence of nth convergents of (2.3), we see that (2.8) shows (as is known [23]) that if $\{K_{n_\nu}(0)\}$ converges to finite t or diverges to $t = \infty$, then $\{K_{n_\nu}(z)\}$ converges to $I_t(z)$, the convergence being uniform on compact sets that exclude the poles of I_t. Also, according to (2.7), if t is finite,

$$\lim_{\nu\to\infty} \frac{P_{n_\nu}(z)}{P_{n_\nu}(0)} = -B(z) + tD(z),$$

while if $t = \infty$,

$$\lim_{\nu\to\infty} \frac{P_{n_\nu}(z)}{N_{n_\nu}(0)} = D(z).$$

In either case, the convergence is uniform on compact sets, so it follows with the aid of Hurwicz' theorem [14, p. 205] that the poles of $L_t(z)$ are precisely the accumulation points of X_a. That is, if $\{K_{n_\nu}(0)\}$ converges to t or diverges to $t = \infty$, then

$$S(\psi_t) = X_a' \tag{2.10}$$

(where the prime indicates the derived set).

In particular, then, for a general indeterminate moment problem, there is an extremal solution whose spectram coincides with the set X' of accumulation points of the zeros of *all* $P_n(x)$ if and only if $\{K_n(0)\}$ converges or else diverges to ∞.

Note that in the symmetric case, $c_n = 0$ in (2.3), we have

$$K_{2n}(0) = 0, \qquad [K_{2n+1}(0)]^{-1} = 0.$$

This confirms, as was shown in [10], that when a symmetric moment problem is indeterminate, there are two extremal solutions, ψ_0 and ψ_∞, whose spectra coincide with the accumulation points of the zeros of the polynomials of even order and of odd order, respectively. Since an extremal solution is determined by a single point in its spectrum [20, p. 60], these are the only extremal solutions whose spectra contain accumulation points of zeros.

Finally, we note that if a subsequence of zeros of the form $\{x_{n_\nu, i_\nu}\}$ converges to a finite limit ξ, then according to (2.7)

$$\lim_{\nu\to\infty} K_{n_\nu}(0) = \lim_{\nu\to\infty} \frac{B_{n_\nu}(x_{n_\nu,i_\nu})}{D_{n_\nu}(x_{n_\nu,i_\nu})} = \frac{B(\xi)}{D(\xi)} = t.$$

Hence $\{K_{n_\nu}(z)\}$ converges to $I_t(z)$ which has ξ as a pole, and the spectrum of the corresponding extremal solution ψ_t is X_a (see (2.9)). In particular, this shows that in order that there be an extremal solution whose spectrum consists of all accumulation points of zeros of all $P_n(x)$, it is necessary and sufficient that a single sequence of zeros of the form $\{x_{n,i_n}\}$ converge to a finite limit. This provides a simple way of concluding the existence of a "best" extremal solution whenever the zeros of the orthogonal polynomials have a finite lower bound.

3. Conditions for Denumerable Spectra

We assume henceforth that the Hamburger moment problem associated with (1.1) is determined. We will obtain two sets of conditions on the coefficients in (1.1) which are sufficient for the corresponding distribution function to have denumerable spectra which are unbounded both above and below. Maintaining the notation (2.1), we first establish the following preliminary result.

Lemma. *If there exists an integer N and a compact set K such that*

$$|\alpha_n(z)| \leqslant \tfrac{1}{4} \qquad \text{for all } n > N \text{ and all } z \in K, \tag{3.1}$$

then $S(\psi) \cap K$ is a finite set.

Proof. The Nth tail of the J-fraction (2.3), viz.,

$$\frac{\lambda_{N+1}\ |}{|\ z - c_{N+1}} - \frac{\lambda_{N+2}\ |}{|\ z - c_{N+2}} - \frac{\lambda_{N+3}\ |}{|\ z - c_{N+3}} - \cdots,$$

is equivalent to

$$\frac{(z-c_N)\,\alpha_{N+1}(z)\ |}{|\ 1} - \frac{\alpha_{N+2}(z)\ |}{|\ 1} - \frac{\alpha_{N+3}(z)\ |}{|\ 1} - \cdots. \tag{3.2}$$

Condition (3.1) ensures that (3.2) converges uniformly on K, and that its limit is an analytic function $G_N(z)$ [23]. It then follows that the J-fraction (2.3) converges to

$$G(z) = \frac{F_N(z) + F_{N-1}(z)\, G_N(z)}{P_N(z) + P_{N-1}(z)\, G_N(z)}$$

at every z in K which is not a zero of the denominator

$$\mathscr{D}_N(z) = P_N(z) + P_{N-1}(z)\, G_N(z).$$

Further, as the limit of a convergent J-fraction, $G_N(z)$ cannot be a rational function so $\mathscr{D}_N(z)$ cannot be identically 0 on K. It follows that $\mathscr{D}_N(z)$ cannot have infinitely many zeros on K; hence G has at most finitely many singularities on K. Thus $S(\psi)$ has at most finitely many points in K.

THEOREM 1. *Let*

(i) $\lim_{n\to\infty} |c_n| = \infty$;

(ii) $\inf_n c_n = -\sup_n c_n = -\infty$;

(iii) $\limsup_{n\to\infty} |\alpha_n(0)| < \frac{1}{4}$.

Then $\sigma = -\tau = -\infty$ *and* $S(\psi)$ *has no finite points of accumulation.*

Proof. Let K be any compact subset of the complex plane. We can write

$$\alpha_n(z) = \alpha_n(0)\{(1 - z/c_n)(1 - z/c_{n+1})\}^{-1}. \tag{3.3}$$

Because of (i) and (iii) it follows that there exists N such that

$$|\alpha_n(z)| < \tfrac{1}{4} \qquad \text{for} \quad n > N \text{ and } z \in K.$$

By the lemma, we now conclude that $S(\psi) \cap K$ is finite for every compact set K, hence $S(\psi)$ has no finite limit points.

Finally, we have generally that

$$\sigma \leqslant \liminf_{n\to\infty} c_n, \qquad \tau \geqslant \limsup_{n\to\infty} c_n \tag{3.4}$$

(see [8, p. 113]) so that (ii) follows immediately.

Although our primary interest in this paper is in the case $\sigma = -\tau = -\infty$, the next theorem can also apply to cases where σ or τ is finite. In fact, the latter may turn out to be the more interesting application.

THEOREM 2. *Let $\mathscr{L}$ denote the set of subsequential limit points (in the extended real number system) of $\{c_n\}$. If*

$$\lim_{n\to\infty} \alpha_n(0) = 0$$

then $\sigma = \inf \mathscr{L}$, $\tau = \sup \mathscr{L}$, and the accumulation points of $S(\psi)$ are a subset of $\mathscr{L}$.

Proof. Let K be any compact set such that $K \cap \mathscr{L} = \varnothing$. Then no subsequence of $\{c_n\}$ can converge to a $z \in K$. Hence there is an index N and a $\delta > 0$ such that for $n > N$, $|1 - z/c_n| > \delta$ and, according to (3.3),

$$|\alpha_n(z)| \leqslant |\alpha_n(0)|\, \delta^{-2} < \tfrac{1}{4} \qquad \text{for all } z \in K.$$

Referring to the lemma once again, we see that $S(\psi) \cap K$ is a finite set for every compact set K which is disjoint from $\mathscr{L}$. It follows that $S(\psi)' \subseteq \mathscr{L}$. The latter when combined with (3.4) then yields the assertions concerning σ and τ.

Remark. When the coefficients are bounded in Theorem 2, we would also have $\lambda_n \to 0$ in which case we could then conclude $S(\psi)' = \mathscr{L}$ [7]. Thus one wonders whether this conclusion holds in the general unbounded case. Without any examples to guide us, it is difficult to conjecture but we speculate that equality does not always hold.

If one adds the rather artificial and fussy condition that a subsequence $\{c_{n_i}\}$ converges to c and, correspondingly, $\{\lambda_{n_i+1} + \lambda_{n_i}\}$ contains a *null* subsequence, then the proof used in [7] can be used with the appropriate subsequence to show that c *is* an accumulation point of $S(\psi)$.

4. REMARKS AND EXAMPLES

The conclusion in Theorem 1 that $S(\psi)$ has no finite accumulation points would follow without the hypothesis (ii). However, dropping (ii) would only add the case where $c_n \to \infty$ and for this the corresponding theorem is known [8, p. 119]. Unfortunately, there are no known examples illustrating Theorem 1. Since it is easy to construct simple recurrence relations satisfying the hypothesis, perhaps others will be encouraged to look for examples. Some likely condidates would seem to be found by taking $(-1)^n c_n$ to be a linear polynomial in n and λ_n to be a positive constant, linear or suitably restricted quadratic polynomial.

As mentioned earlier, our initial motivation for obtaining Theorem 2 was to apply to the case $\sigma = -\infty$, $\tau = \infty$. However, in many respects, the more interesting application is probably to the case where σ is finite and $\tau = \infty$.

Once again, we know of no explicit examples but perhaps some can be found once a general situation is described.

For example, in the simplest nontrivial case, $\mathscr{L} = \{c, \infty\}$, relatively simple recurrence formulas can be written with a great deal of flexibility. For we can take an arbitrary positive null chain sequence $\{a_n\}$. (For example, one can take $0 < a_n \leqslant (n+1)^{-1}$.) Then choose $\{c_n\}$ so that

$$c_{2n} \to c \text{ (finite)} \qquad \text{and} \qquad c_{2n+1} \to \infty$$

and such that

$$\sum (a_{2n+1} c_{2n+1})^{-1/2} = \infty. \tag{4.1}$$

Finally define

$$\lambda_{n+1} = a_n c_n c_{n+1}.$$

Then $\Sigma \lambda_n^{-1/2} = \infty$ so the corresponding Hamberger moment problem is determined according to Carleman's criterion [20]. Further, $\{\alpha_n(0)\} = \{a_n\}$ is a chain sequence so $S(\psi) \subseteq [0, \infty)$. Finally, by Theorem 2, $S(\psi)' = \{c, \infty\}$.

For example, we can take $c_{2m-1} = 1$ and arbitrary positive c_{2m} such that

$$c_{2m} \to \infty \qquad \text{and} \qquad \sum (m/c_{2m})^{1/2} = \infty,$$

$$\lambda_{2m} = c_{2m}/2m, \qquad \lambda_{2m+1} = c_{2m}/(2m+1).$$

Then

$$\alpha_n(0) = \frac{1}{n+1} = \left(1 - \frac{n-1}{n}\right) \frac{n}{n+1}.$$

Thus $\{\alpha_n(0)\}$ is a chain sequence whose minimal parameters are $m_n = n/(n+1)$. Since $m_n \to 1$, it follows that they are also maximal parameters and this means that the true interval of orthogonality is $[0, \infty)$. And we have the single *finite* spectral limit point $\sigma = 1$.

We remark that for the corresponding kernel polynomials $Q_n(x)$ (the polynomials which are orthogonal with respect to $x d\psi(x)$), the corresponding recurrence relation is

$$Q_n(x) = (x - d_n)\, Q_{n-1}(x) - \nu_n Q_{n-2}(x),$$

where

$$d_{2m-1} = c_{2m}, \qquad d_{2m} = 1, \qquad \nu_n = \frac{n-1}{n(n+1)} c_{2[(m+1)/2]}.$$

Perhaps a fortuitous choice of the c_{2m} might lead to a case where it would be possible to determine the corresponding orthogonality relations explicitly.

REFERENCES

1. N. I. AHIEZER AND M. G. KREIN, "Some Questions in the Theory of Moments," Translations of Math. Monographs, Vol. 2, Amer. Math. Soc., Providence, R. I., 1962.
2. O. BLUMENTHAL, Uber die entwicklung einer willkürlichen Funktion nach den Nennern des Kettenbruches für $\int_{-\infty}^{0}[\phi(\xi)/(z-\xi)]\,d\xi$, Dissertation, Göttingen, 1898.
4. K. M. CASE, Orthogonal polynomials revisited, *in* "Theory and Application of Special Functions" (R. Askey, Ed.), Academic Press, New York, 1975.
5. T. S. CHIHARA, Orthogonal polynomials whose zeros are dense in intervals, *J. Math. Anal. Appl.* **24** (1968), 362–371.
6. T. S. CHIHARA, Indeterminate Hamburger moment problems, *Pacific J. Math.* **27** (1968), 475–484.
7. T. S. CHIHARA, The derived set of the spectrum of a distribution function, *Pacific J. Math.* **35** (1970), 571–574.
8. T. S. CHIHARA, "An Introduction to Orthogonal Polynomials," Gordon & Breach, New York, 1978.
9. T. S. CHIHARA, Orthogonal polynomials whose distribution functions have finite point spectra, *SIAM J. Math. Anal.* **11** (1980), 358–364.
10. T. S. CHIHARA, Indeterminate symmetric moment problems, *J. Math. Anal. Appl.* **85** (1982), 331–346.
11. T. S. CHIHARA, Spectral properties of orthogonal polynomials on unbounded sets, *Trans. Amer. Math. Soc.* **270** (1982), 623–639.
12. J. GERONIMO AND K. M. CASE, Scattering theory and polynomials orthogonal on the real line, *Trans. Amer. Math. Soc.* **258** (1980), 467–494.
13. J. GERONIMO AND P. G. NEVAI, Necessary and sufficient conditions relating the coefficients in the recurrence formula to the spectral function for orthogonal polynomials, *SIAM J. Math. Anal.* **14** (1983), 622–637.
14. E. HILLE, "Analytic Function Theory," Vol. II, Ginn, Boston, 1962.
15. D. P. MAKI, A note on recursively defined orthogonal polynomials, *Pacific J. Math.* **28** (1969), 611–613.
16. A. MÁTÉ AND P. G. NEVAI, Orthogonal polynomials and absolutely continuous measures, to appear.
17. P. G. NEVAI, "Orthogonal Polynomials," Mem. Amer. Math. Soc., No. 213, 1979.
18. P. G. NEVAI, Distribution of zeros of orthogonal polynomials, *Trans. Amer. Math. Soc.* **249** (1979), 341–361.
19. P. G. NEVAI, Orthogonal polynomials defined by a recurrence relation, *Trans. Amer. Math. Soc.* **250** (1980), 369–384.
20. J. SHOHAT AND J. D. TAMARKIN, "The Problem of Moments," Math. Surveys, No. 1, Amer. Math. Soc., Providence, R.I., 1943, 1950.
21. T. J. STIELTJES, "Recherches sur les fractions continues," Oeuvres, Tome II, Noordhoff, Groningen, 1918.
22. M. H. STONE, "Linear Transformations in Hilbert Space," Colloq. Publ. Vol. 15, Amer. Math. Soc., Providence, R. I., 1932.
23. H. S. WALL, "Analytic Theory of Continued Fractions," Van Nostrand, New York, 1948.

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